ON THE APPLICATION OF COMPLEX VARIABLES IN PLANE PLASTIC DEFORMATIONS

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The conditions of the stress function being real, applicable to the solution of inverse problems, are derived. A few such solutions are given as illustrative examples. The conditions of the stress function being biharmonic in the plastic range are also determined.

1. Consider a homogeneous isotropic material in the plastic state. The stress components satisfy the equilibrium conditions

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}=0, \qquad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}=0 \tag{1.1}$$

and the Huber-Von Mises (or St. Venant-Tresca) plasticity conditions

$$(\sigma_x-\sigma_y)^2+4\tau_{xy}^2=4k^2 \tag{1.2}$$

where k is the yield limit in shear [1]. We introduce the stress function F:

$$\sigma_x=\frac{k}{2}\frac{\partial^2F}{\partial y^2}, \qquad \sigma_y=\frac{k}{2}\frac{\partial^2F}{\partial x^2}, \qquad \tau_{xy}=-\frac{k}{2}\frac{\partial^2F}{\partial x\partial y} \tag{1.3}$$

Equation (1.1) is satisfied identically, and (1.2) is left for the determination of function F. This equation can be represented in the complex variable $z = x + iy$ as

$$\frac{\partial^2F}{\partial z^2}\frac{\partial^2F}{\partial\bar{z}^2}=1 \tag{1.4}$$

From physical considerations F has clearly to be real, i.e.

$$F(z,\bar{z})=\overline{F(z,\bar{z})} \tag{1.5}$$

Now (1.4) can be written as

$$\frac{\partial^2F}{\partial z^2}=\exp[i\Theta] \tag{1.6}$$

where $\Theta = \Theta(z, \bar{z})$ is a real function. Integration of (1.6) results in

$$F(z, \bar{z}) = \int_{z_0}^{z} d\eta \int_{\eta_0}^{\eta} \exp[i\Theta(\xi, \bar{z})]\, d\xi + \overline{z\varphi(z)} + \overline{\chi(z)} \tag{1.7}$$

Theorem 1. The necessary and sufficient condition for the function $F(z, \bar{z})$ defined by (1.7) to be real is

$$\frac{\partial^2 e^{i\Theta}}{\partial \bar{z}^2} = \frac{\partial^2 e^{-i\Theta}}{\partial z^2} \tag{1.8}$$

Proof. Applying the operator

$$\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} \tag{1.9}$$

to the functions $F(z, \bar{z})$ and $\overline{F(z, \bar{z})}$ and taking into account (1.5) we obtain the proof of the necessity of the condition (1.8). To prove that this condition is also sufficient we write (1.8) as

$$\frac{\partial^4 F}{\partial z^2 \partial \bar{z}^2} = \frac{\partial^4 \bar{F}}{\partial z^2 \partial \bar{z}^2}$$

The operator (1.9) is biharmonic, and consequently F can differ from $\bar{F}$ by a biharmonic additive term which can be selected so that $F = \bar{F}$. Thus the theorem is proved.

The condition (1.8), for practical purposes, may be conveniently represented in the Cartesian coordinates

$$\cos\Theta\left(\frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial y^2} - 2\frac{\partial\Theta}{\partial x}\frac{\partial\Theta}{\partial y}\right) + \sin\Theta\left(-2\frac{\partial^2\Theta}{\partial x\,\partial y} - \left(\frac{\partial\Theta}{\partial x}\right)^2 + \left(\frac{\partial\Theta}{\partial y}\right)^2\right) = 0 \tag{1.10}$$

or in the polar coordinates (r, ϕ)

$$\cos(\Theta + 2\varphi)\left(\frac{\partial^2\Theta}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\Theta}{\partial\varphi^2} - \frac{2}{r}\frac{\partial\Theta}{\partial r}\frac{\partial\Theta}{\partial\varphi} - \frac{1}{r}\frac{\partial\Theta}{\partial r}\right) +$$
$$+ \sin(\Theta + 2\varphi)\left(\frac{2}{r^2}\frac{\partial\Theta}{\partial\varphi} + \frac{1}{r^2}\left(\frac{\partial\Theta}{\partial\varphi}\right)^2 - \frac{2}{r}\frac{\partial^2\Theta}{\partial r\partial\varphi} - \left(\frac{\partial\Theta}{\partial r}\right)^2\right) = 0 \tag{1.11}$$

Introducing the notation $\exp[i\Theta] = u + iv$, (1.8) can be represented as

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = \frac{2v}{\sqrt{1 - v^2}}\left(v\frac{\partial^2 v}{\partial x\,\partial y} + \frac{1}{1 - v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) \tag{1.12}$$

Theorem 2. A particular solution $\Theta = \Theta(z, \bar{z})$ of (1.8), which does not contain arbitrary parameters, determines the stress components within a constant hydrostatic pressure.

Proof. Theorem 1 and Expression (1.7) show that the solution $\Theta = \Theta(z, \bar{z})$ determines a real stress function $F(z, \bar{z})$ within an additive term $pz\bar{z}$ (p = const) which represents constant hydrostatic pressure.

Special interest attaches to stress functions which satisfy the biharmonic equation in the plastic range.

Theorem 3. The necessary and sufficient condition for the stress function F to be a biharmonic function is that the function $\Theta = \Theta(z, \bar{z})$ satisfies the following system of equations:

$$\frac{\partial^2\Theta}{\partial x^2} - \frac{\partial^2\Theta}{\partial y^2} - 2\frac{\partial\Theta}{\partial x}\frac{\partial\Theta}{\partial y} = 0, \qquad \left(\frac{\partial\Theta}{\partial y}\right)^2 - \left(\frac{\partial\Theta}{\partial x}\right)^2 - 2\frac{\partial^2\Theta}{\partial x\,\partial y} = 0 \qquad (1.13)$$

Proof. If F, represented by (1.7), is a biharmonic function, then in conjunction with (1.6) the following holds:

$$\frac{\partial^4 F}{\partial z^2\,\partial\bar{z}^2} = 0, \quad \text{or} \quad \frac{\partial^2 e^{i\Theta}}{\partial\bar{z}^2} = 0 \qquad (1.14)$$

Representation of this relationship in Cartesian coordinates leads to (1.13), and thus to the proof of the necessity condition.

Inverse arguments lead from (1.13) to (1.14) and to the proof of the sufficiency condition.

It is obvious that the solutions of (1.13) are also solutions of (1.8) or (1.10).

The class of solutions of (1.13) is apparently very small, but it is not empty. As an example we propose the following solutions:

1. Θ = const, corresponding to a uniform stress field;

2. $\Theta = -2\tan^{-1}(y/x)$ + const. This case was utilized by Galin [2].

Note 1. Equations (1.10) and (1.12) possess the following features. If $\Theta = \Theta(x, y)$ is a solution of (1.10), then $\Psi_1 = -\Theta(-x, y)$ and $\Psi_2 = -\Theta(x, -y)$, and consequently $\Psi_3 = \Theta(-x, -y)$ are also solutions. If $w = u + iv$ is a solution of (1.12), then $w_1 = u_1 + iv_1$ is also a solution of the same equation.

Note 2. If some solution of (1.8) is known, then it is not necessary to find the stress function from (1.7) and to calculate the stresses from this function. It is easier to utilize the equilibrium equation (1.1) and the fact that Θ determines uniquely the difference between the normal and shear stresses.

2. We shall now determine a few particular solutions of the equilibrium equations in the plastic range by solving (1.10) and (1.11).

1. We seek a solution in the form $\Theta = \Theta(y)$. Equation (1.10) can thus be written

$$\frac{d^2\Theta}{dy^2} - \tan\Theta\left(\frac{d\Theta}{dy}\right)^2 = 0$$

Solving this equation we find $\Theta = \sin^{-1}(Ay + B)$. This corresponds to a case of a strip compressed by two rough plates [1,3]

$$\sigma_x = p - Ax - 2\sqrt{1-(Ay+B)^2}, \qquad \sigma_y = p - Ax, \qquad \tau_{xy} = Ay + B \tag{2.1}$$

where the stresses are represented relative to the magnitude k. A solution in the form $\Theta = \Theta(\alpha x + \beta y)$ reduces to the previous one by rotating the coordinate axes.

2. Seeking a solution in the form $\Theta = -2\phi + R(r)$ reduces Equation (1.12) to

$$\frac{d^2R}{dr^2} - \tan R\left(\frac{dR}{dr}\right)^2 + \frac{3}{r}\frac{dR}{dr} = 0$$

Solving this equation we find

$$R = \arcsin\left(a - \frac{b}{r^2}\right), \qquad \text{or} \qquad \Theta = -2\varphi + \sin^{-1}\left(a - \frac{b}{r^2}\right) \tag{2.2}$$

and the corresponding stress components are

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = -2a\varphi + \sqrt{1-a^2}\ln\left[\sqrt{1-a^2}\sqrt{(1-a^2)r^4 + 2abr^2 - b^2} + \right.$$
$$\left. + (1-a^2)r^2 + ab\right] + a\sin^{-1}\left(a - \frac{b}{r^2}\right) \mp \frac{1}{r^2}\sqrt{(1-a^2)r^4 + 2abr^2 - b^2} + p \tag{2.3}$$
$$\tau_{r\varphi} = a - \frac{b}{r^2}$$

The displacement components found from a simplified Hencky-Von Mises theory [1] and independent of the angle ϕ are

$$u_r = \frac{c}{r^2}, \qquad u_\varphi = dr - \frac{c}{br}\sqrt{(1-a^2)r^4 + 2abr^2 - b^2} \tag{2.4}$$

Here $b \neq 0$. If $b = 0$ then

$$u_r = \frac{c}{r^2}, \qquad u_\varphi = dr - \frac{ac}{\sqrt{1-a^2}}\frac{1}{r} \tag{2.5}$$

The solution represented by (2.3) and (2.4) is the most general solution among all known closed-form solutions. By specializing the values of the parameters entering into this solution, we obtain a series of particular solutions.

a) For $a = 0$, $b = 0$, we obtain

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = \ln r^2 \mp 1 + p, \qquad \tau_{r\varphi} = 0 \tag{2.6}$$

which represents an axisymmetrical stress field [1].

b) For $a = 1$, $b = 0$, we obtain

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = -2\varphi + p, \qquad \tau_{r\varphi} = 1 \tag{2.7}$$

which is a solution for a plastic wedge with uniformly loaded faces.

c) For $b = 0$ the superposition of the two previous cases results in (cf. [1])

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = -2a\varphi + \sqrt{1-a^2}\,(\ln r^2 \mp 1) + p, \qquad \tau_{r\varphi} = a \tag{2.8}$$

d) For $a = 0$, $b = -c$ there results a most general case of axisymmetric loading [1]:

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = \varepsilon\left[\ln(\sqrt{r^4-c^2} + r^2) \mp \frac{1}{r^2}\sqrt{r^4-c^2}\right] + p, \qquad \tau_{r\varphi} = \frac{c}{r^2} \qquad (\varepsilon = \mp 1) \tag{2.9}$$

e) Letting $a = 1$, $b = c$ we obtain a new solution of a problem presented in [3]:

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = -2\varphi + 2\tan^{-1}\frac{\sqrt{2cr^2-c^2}}{c} \mp \frac{\sqrt{2cr^2-c^2}}{r^2} + p, \qquad \tau_{r\varphi} = 1 - \frac{c}{r^2} \tag{2.10}$$

with the corresponding displacements

$$u_r = \frac{c_1}{r^2}, \qquad u_\varphi = dr - \frac{c_1}{cr}\sqrt{2cr^2-c^2} \tag{2.11}$$

Along the contours $r = r_0$ the stresses and displacements are

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = p_1 - 2\varphi \mp p_2, \qquad \tau_{r\varphi} = 1 - \frac{c}{r_0^2} \qquad (p_1, p_2 = \mathrm{const})$$

$$u_r = \frac{c_1}{r_0^2}, \qquad u_\varphi = dr_0 - \frac{c_1}{cr_0}\sqrt{2cr_0^2-c^2}$$

Letting $d = 0$ and $r \to \infty$ we obtain

$$\left.\begin{matrix}\sigma_r\\ \sigma_\varphi\end{matrix}\right\} = -2\varphi + \pi + p, \qquad \tau_{r\varphi} = 1, \qquad u_r = 0, \qquad u_\varphi = -c_1\sqrt{\frac{2}{c}} \tag{2.12}$$

BIBLIOGRAPHY

1. Sokolovskii, V.V., *Teoriia plastichnosti* (*Theory of Plasticity*). Izd-vo Akad. Nauk SSSR, 1940.

2. Galin, L.A., Ploskaia uprugo-plasticheskaia zadacha (Plane elasto-plastic problem). *PMM* Vol. 10, No. 3, 1946.

3. Dobrovolskii, V.L., Zadacha o ploskoi deformatsii ideal'no plasticheskogo tela v kompleksnykh peremennykh (On a problem of plane deformations of an ideally plastic body in complex variables). *PMM* Vol. 24, No. 2, 1960.

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